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Special Hperbolic Type Approximation for Solving of 3-D Two Layer Stationary Diffusion Problem

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***Abstract*—In this paper we examine the conservative averaging method (CAM) along the vertical z-coordinate for solving the 3-D boundary-value 2 layers diffusion problem. The special parabolic and hyperbolic type approximation (splines), that interpolate the middle integral values of piece-wise smooth function, is investigated. With the help of these splines the problems of mathematical physics in 3-D with respect to one coordinate are reduced to problems for system of equations in 2-D in every layer. This procedure allows reduce also the 2-D problem to a 1-D problem and the solution of the approximated problem can be obtained analytically. As the practical application of the created mathematical model, we are studying the calculation of the concentration of heavy metal calcium (Ca) in a two-layer peat block.**

Keywords—conservative averaging method, finite-difference method, diffusion problem, special splines.

Introduction

The boundary value problems (BVP) described by PDE with piece-wise coefficients in multi-layered domains are currently the subject of studies [4]. The interest is often caused by problems in itself, but even more interesting are their solutions: mainly numerical ones, because analytical solutions can only be obtained in the ordinary sense (without changing the number of dimensions of the boundary-value problem) in the simplest cases [2]. The article deals with a universal method for solving the second order of partial differential equations – a consider conservative averaging method (CAM), the essence of which is a reduction in the number of dimensions of a given BVP, with a view to obtaining analytical expressions (formulas) of the solution. The further solution of the BVP includes a repeated reduction in the number of dimensions or applying of numerical methods to solve the acquired BVP. The unknown function is replaced by the approximated solution – the special spline with two different functions, which interpolate the middle integral values of piece-wise smooth function. The functions of the hyperbolic type spline are created and used with parameters that have to be chosen in the appropriate way to decrease the error of approximation of the solution. It should be noted that, in limit case when the parameters of spline function tends to zero we have the integral parabolic spline, obtained from A.Buikis [3]. The 2-D boundary value problem obtained by the conservative averaging method (CAM) was solved numerically using the finite-difference method in the case of parabolic and hyperbolic splines. A test example, a solution to the given 3-D boundary value problem, was created for numerical approbation of the averaging method, where the unknown function was a solution to the corresponding 1-D boundary value problem for two ODEs. The solution of ODEs' could be obtained both analytically (exact solution) and numerically by the averaging method using parabolic type and hyperbolic type splines. This in turn allowed a comparison of the analytical solution with the obtained numerical solutions.

Materials and methods

1.The Mathematical Model

The process of diffusion is considered in 3-D parallelepiped

$$\Omega=\{(x,y,z): 0\le x\le l, 0\le y\le L, 0\le z\le Z\}$$

. The domain Ω consists of two layer medium. We will consider the stationary 3-D problem of the linear diffusion theory for multilayered piece-wise homogenous materials of N layers in the form $\Omega_i=\{(x,y,z): x\in(0,l)\ y\in(0,L)\ z\in(z_{i-1},x_i)\}, i=\overline{1,N}$ where $H_i=z_i-z_{i-1}$ is the height of layer

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$\Omega_i, z_0 = 0, z_N = Z$. The distribution of concentrations $c_i = c_i(x,y,z)$ in every layer Ω_i at the point $(x,y,z) = \Omega_i$ should be calculated by solving the following partial differential equation(PDE):

$$D_{ix}\partial^2 c_i/\partial x^2 + D_{iy}\partial^2 c_i/\partial y^2 + D_{iz}\partial^2 c_i/\partial z^2 + f_i(x,y,z) = 0, \quad (1.1)$$

D_{ix}, D_{iy}, D_{iz} are constant diffusion coefficients, $c_i = c_i(x,y,z)$ - the concentrations functions in every layer, $f_i(x,y,z)$ - the fixed sources functions. The values c_i and the flux functions $D_{iz}\partial c_i/\partial z$ must be continuous on the contact lines between the layers $z = z_i, i = \overline{1, N-1}$:

$$c_i|_{z_i} = c_{i+1}|_{z_i}, \quad D_{iz}\partial c_i/\partial z|_{z_i} = D_{(i+1)z}\partial c_{(i+1)}/\partial z|_{z_i} \quad (1.2)$$

where $i = \overline{1, N-1}$. The layered material is bounded above and below with the plane surfaces $z = 0, z = Z$ with fixed boundary conditions in following form:

$$D_{1z}\partial c_1(x,y,0)/\partial z = 0, \quad c_N(x,y,Z) = C_a(x,y), \quad (1.3)$$

or $\quad c_1(x,y,0) = C_0(x,y), \quad c_N(x,y,Z) = C_a(x,y), \quad (1.4)$

$C_0(x,y)$, $C_a(x,y)$ are the given concentration-functions.

We have two forms of fixed boundary conditions in the x,y directions: 1) The periodical conditions by $x = 0, x = l$ (1.5):
$c_i(0,y,z) = c_i(l,y,z)$, $\partial c_i(0,y,z)/\partial x = \partial c_i(l,y,z)/\partial x$,

2) The symmetrical conditions by $y = 0, y = L$ (1.6): $\partial c_i(x,0,z)/\partial y = \partial c_i(x,L,z)/\partial y = 0$.

We will use the CAM and the finite difference (FD) method to solve the problem (1.1)-(1.6). These procedures allow reduce the 3-D problem to some 2-D boundary-value problem (BVP) for the system of partial differential equations with circular matrix in the x-direction.

2.The conservative averaging method with parabolic splines

The equation of (1.1) are averaged along the heights H_i of layers Ω_i and quadratic integral splines along z coordinate in the following form one used [3]

$$c_i(x,y,z) = C_i(x,y) + m_i(x,y)(z - \bar{z}_i) + e_i(x,y)G_i\left((z-\bar{z}_i)^2/H_i^2 - 1/12\right), \quad (2.1)$$

$G_i = H_i/D_{iz}$, $\bar{z}_i = (z_{i-1} + z_i)/2$. m_i, e_i, C_i - the unknown coefficients of the spline-function,

$C_i(x,y) = H_i^{-1}\int_{z_{i-1}}^{z_i} c_i(x,y,z)dz$ - the average values of $c_i, i = \overline{1,N}$.

After averaging the system (1.1) along every layer Ω_i we obtain

$$D_{ix}\partial^2 C_i/\partial x^2 + D_{iy}\partial^2 C_i/\partial y^2 + 2H_i^{-1}e_i + F_i(x,y) = 0, \quad (2.2)$$

$F_i(x,y) = H_i^{-1}\int_{z_{i-1}}^{z_i} f_i(x,y,z)dz$ - the average values of $f_i, i = \overline{1,N}$. From (1.1), (2.1) using boundary conditions (1.3) we can determine the unknown functions m_i, e_i. Therefore, from (2.2) we obtain the system of N partial differential equations (PDE), where the boundary conditions for C_i are determined from (1.4)-(1.5) in the x,y-directions for averaged values

$$C_i(0,y) = C_i(l,y), \quad \partial C_i(0,y)/\partial x = \partial C_i(l,y)/\partial x \quad (2.3)$$

$$\partial C_i(x,0)/\partial y = \partial C_i(x,L)/\partial y = 0. \quad (2.4)$$

In the case $N = 2$ (two layers) we have (2.5):

$$e_1 = \left(9C_2 - (12 + 3/k)C_1 - 3C_a + (6 + 3/k)C_0\right)/(2G_2 + 2G_1),$$

$$e_2 = \left(9C_1 - (12 + 3k)C_2 + (6 + 3k)C_a - 3C_0\right)/(2G_2 + 2G_1),$$

$$k = G_1/G_2.$$

We have from (2.2) the following system of two PDE

$$\begin{cases} D_{1x}\partial^2 C_1(x,y)/\partial x^2 + D_{1y}\partial^2 C_1(x,y)/\partial y^2 + \\ 2H_1^{-1}e_1(x,y) + F_1(x,y) = 0, \\ D_{2x}\partial^2 C_2(x,y)/\partial x^2 + D_{2y}\partial^2 C_2(x,y)/\partial y^2 + \\ 2H_2^{-1}e_2(x,y) + F_2(x,y) = 0. \end{cases} \quad (2.6)$$

After resolving (2.6), the concentration functions $c_i = c_i(x,y,z)$ shall be obtained

$$c_1(x,y,z) = C_1(x,y) + m_1(x,y)(z - H_1/2) + e_1(x,y)G_1\left((z - H_1/2)^2/H_1^2 - 1/12\right), z \in [0, H_1],$$

$$c_2(x,y,z) = C_2(x,y) + m_2(x,y)(z - (2H_1 + H_2)/2) + e_2(x,y)G_2\left((z - (2H_1 + H_2)/2)^2/H_2^2 - 1/12\right), z \in [H_1, Z].$$

From (1.3) it is obtained

$$m_1(x,y) = e_1(x,y)/(3D_{1z}) + 2(C_1(x,y) - C_0(x,y))/H_1,$$

$$m_2(x,y) = -e_2(x,y)/(3D_{2z}) + 2(C_a(x,y) - C_2(x,y))/H_2$$

3.The conservative averaging method with hyperbolic type splines in 2 layers

The equation of (1.1) is averaged along the heights H_i of layers Ω_i using the hyperbolic type splines. Applying averaged method with respect to z we use the approximate solution with two fixed parametrical functions $f_{i1}, f_{i2}, i = \overline{1,2}$

$c_i(x,y,z) = C_i(x,y) + m_i(x,y) f_{i1}(z - \bar{z}_i) + e_i(x,y) + f_{i2}(z - \bar{z}_i)$,

$C_i(x,y) = H_i^{-1} \int_{z_{i-1}}^{z_i} c_i(x,y,z) dz$ - the averaged values, according to the definition of the spline function

$$\int_{z_{i-1}}^{z_i} f_{i1}(z) dz = \int_{z_{i-1}}^{z_i} f_{i2}(z) dz = 0,$$

$$\bar{z}_i = (z_{i-1} + z_i)/2, z \in [z_{i-1}, z_i],$$

$$f_{iz1} = \frac{0.5 H_i \sinh(a_i(z - \bar{z}_i))}{\sinh(0.5 a_i H_i)},$$

$$f_{iz2} = \frac{\cosh(a_i(z - \bar{z}_i)) - A_i}{8 \sinh^2(0.5 a_i H_i)},$$

$$A_i = \frac{0.5 \sinh(0.5 a_i H_i)}{a_i L/2}, i = \overline{1,2}.$$

$a_i > 0$ are fixed parameters (unknown). It should be noted if parameters a_i tend to zero then the integral parabolic spline from [3] is obtained in the limit case. The unknown functions $m_i(x,y)$, $e_i(x,y)$ we can determined from boundary conditions at $z = 0, z = Z$:

$$d_{iz} = 0.5 H_i a_i \coth(0.5\, a_i H_i)$$
$$k_{iz} = 0.25 a_i \coth(0.25\, a_i H_i)$$
$$C_1 - 0.5 m_1 H_1 + e_1 b_{1z} = C_0$$
$$C_2 + 0.5 m_2 H_2 + e_2 b_{2z} = C_a$$
$$b_{iz} = \frac{\cosh(0.5 a_i H_i) - A_i}{8 \sinh^2(0.25 a_i H_i)}$$
$$D_{1z}(m_1 d_{1z} + e_1 k_{1z}) = D_{2z}(m_2 d_{2z} - e_2 k_{2z})$$
$$C_1 + 0.5 m_1 H_1 + e_1 b_{1z} = C_2 - 0.5 m_2 H_2 + e_2 b_{2z},$$
$$m_1 = 2(C_1 + b_{1z} e_1)/H_1.$$

Thus we have a system of 2 algebraic equations for determining $e_i, i = \overline{1,2}$:

$b_{11} e_1 + b_{12} e_2 = b_3 C_a + b_4 C_0 - b_3 C_2 - b_4 C_1$, $b_{21} e_1 + b_{22} e_2 = 2C_2 - 2C_1 - C_a + C_0$, where $b_{11} = b_4 b_{1z} + k_{1z}$, $b_{12} = b_3 b_{2z} + D_{21} k_{2z}$, $b_{21} = 2 b_{1z}$, $b_{22} = -2 b_{2z}$, $b_3 = 2 D_{21} d_{2z}/H_2$, $b_4 = 2 d_{1z}/H_1$, $D_{21} = D_{2z}/D_{1z}$.

The solution for e_1, e_2 is:

$e_1 = b_{17} C_1 + b_{18} C_2 + d_4 C_a + d_6 C_0$, $e_2 = b_{27} C_1 + b_{28} C_2 + d_5 C_a + d_7 C_0$, where $b_{17} = (-b_4 b_{22} + 2 b_{12})/\det$, $b_{18} = (-b_3 b_{22} - 2 b_{12})/\det$, $b_{27} = (b_4 b_{21} - 2 b_{11})/\det$, $b_{28} = (b_3 b_{21} + 2 b_{11})/\det$, $d_4 = (b_3 b_{22} + b_{12})/\det$, $d_6 = (b_4 b_{22} - b_{12})/\det$, $d_5 = -(b_3 b_{21} + b_{11})/\det$, $d_7 = (-b_4 b_{21} + b_{11})/\det$, $\det = b_{11} b_{22} - b_{21} b_{12}$.

The 2-D boundary-value problem is in following form:

$$\begin{cases} \frac{\partial}{\partial x}\left(D_{1x} \frac{\partial C_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_{1y} \frac{\partial C_1}{\partial y}\right) + b_5 e_1 + F_1 = 0, \\ \frac{\partial}{\partial x}\left(D_{2x} \frac{\partial C_2}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_{2y} \frac{\partial C_2}{\partial y}\right) + b_6 e_2 + F_2 = 0, \\ \frac{\partial c_{iz}(0,y)}{\partial x} = \frac{\partial c_{iz}(x,0)}{\partial y} = 0, c_{iz}(l,y) = c_{iz}(x,L) = 0, \end{cases} \quad (3.1)$$

where $b_5 = 2 D_{1z} k_{1z}/H_1$, $b_6 = 2 D_{2z} k_{2z}/H_2$.

4.The finite-difference method for two layers with parabolic type splines

We consider an uniform grid $(N_x \times (N_y + 1))$ [7],

$$\omega_h = \left\{ \begin{array}{l} (x_i, y_j), x_i = h_x, y_j = (j-1) h_y, i = \overline{1, N_x}, \\ j = \overline{1, N_y + 1}, N_x h_x = l, N_y h_y = L \end{array} \right\}.$$

Subscripts (i,j) refer to x, y indices; the mesh spacing in the x_i, y_j directions is h_x, h_y. For two layers ($N = 2$) we can the PDEs (2.6) rewritten in following vector form:

$$D_x \partial^2 C/\partial x^2 + D_y \partial^2 C/\partial y^2 - AC + F = 0, \quad (4.1)$$

where D_x, D_y are the 2 order diagonal matrices with elements D_{1x}, D_{2x} and D_{1y}, D_{2y}. A is the matrix of second order, C is the 2nd order vectors-column with elements C_1, C_2 and F is the 2 order vectors-column with following elements :

$$\begin{pmatrix} F_1 - ((6 + 3k^{-1}) C_0 - 3 C_a)/(H_1 d_1) \\ F_1 + ((6 + 3k) C_a - 3 C_0)/(H_2 d_1) \end{pmatrix}^T,$$

$$A = \frac{1}{d_1} \begin{pmatrix} (12 + 3k^{-1})/H_1 & -9/H_1 \\ -9/H_2 & (12 + 3k)/H_2 \end{pmatrix},$$

$(d_1 = G_2 + G_1)$. The equation (4.1) with periodical conditions (2.4) for vector function C in the uniform grid (x_i, y_j) is replaced by vector difference equations of second order approximation in 3- point stencil [1]:

$$\boldsymbol{A}\ W_{j-1} - \boldsymbol{C}\ W_j + \boldsymbol{B}\ W_{j+1} + F_j = 0, \quad (4.2)$$

where W_j are vectors-column $(W_j \approx (C_{1,j}, C_{2,j}, ..., C_{N_x,j})^T)$, F_j are vectors-column with elements $(F_{1,j}, F_{2,j}, ..., F_{N_x,j})^T, \overline{j = 2, N_y}$, $AA, CC, BB = AA$ are the block- matrices of second order with the elements of the circular symmetric matrix with $N_x = M$ -order in the following form (it is possible to define the circular matrix with the first row that is in the form $A = \lfloor a_1, a_2, ..., a_M \rfloor$):

$$\boldsymbol{A} = \begin{pmatrix} [D_{1y}/h_y^2, 0, ..., 0] & 0 \\ 0 & [D_{2y}/h_y^2, 0, ..., 0] \end{pmatrix},$$

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 \\ \boldsymbol{c}_3 & \boldsymbol{c}_4 \end{pmatrix},$$

where the circular matrices $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4$ are

$$cc_1 = \begin{bmatrix} 2(D_{1x}/h_x^2 + D_{1y}/h_y^2) + (12 + 3k^{-1})/(H_1 d_1), -D_{1x}/h_x^2, \\ 0, ..., 0, -D_{1x}/h_x^2 \end{bmatrix}$$

$$cc_2 = [-9/(H_1 d_1), 0, ..., 0], \quad cc_3 = [-9/(H_2 d_1), 0, ..., 0],$$

$$cc_4 = \begin{bmatrix} 2(D_{2x}/h_x^2 + D_{2y}/h_y^2) + (12 + 3k)/(H_2 d_1), \\ -D_{2x}/h_x^2, 0, ..., 0, -D_{2x}/h_x^2 \end{bmatrix}.$$

The boundary conditions (2.4) are replaced by difference equations of first order approximation:

$$C(x, h_y) = C(x, 0) + O(h_y^2), \quad C(x, L) = C(x, L - h_y) + O(h_y^2).$$

5.The finite-difference method for two layers with hyperbolic type splines

The vector F and matrix A in (4.1) are

$$F_1 + b_5 d_4 C_a + b_5 d_6 C_0, \; F_2 + b_6 d_5 C_a + b_6 d_7 C_0,$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \text{ where}$$

$$a_{11} = -b_5 b_{17}, \; a_{12} = -b_5 b_{18}, \; a_{21} = -b_6 b_{27},$$
$$a_{22} = -b_6 b_{28}.$$

The circular matrices $\boldsymbol{c}_i$ are $a_{22} = -b_6 b_{28}$

$$cc_1 = \begin{bmatrix} 2(D_{1x}/h_x^2 + D_{1y}/h_y^2) + a_{11}, \\ -D_{1x}/h_x^2, 0, ..., 0, -D_{1x}/h_x^2 \end{bmatrix},$$

$$cc_2 = [a_{12}, 0, ..., 0], \quad cc_3 = [a_{21}, 0, ..., 0],$$

$$cc_4 = \begin{bmatrix} 2(D_{2x}/h_x^2 + D_{2y}/h_y^2) + a_{22}, \\ -D_{2x}/h_x^2, 0, ..., 0, -D_{2x}/h_x^2 \end{bmatrix}.$$

We use $a_1 = 15, a_2 = 13$.

6. The numerical methods

The vectors-column W_j from (4.2) is calculated on Thomas algorithm [9] in the matrix form using MATLAB.

$$W_j = X_j W_{j+1} + Y_j = 0, \; j = N_y(-1)1, \quad (6.1)$$

where X_j, Y_j are corresponding matrices and vectors, obtaining of following expressions (6.2):

$$X_j = (\boldsymbol{C}_j - \boldsymbol{A}_j X_{j-1})^{-1} \boldsymbol{B}_j,$$
$$Y_j = (\boldsymbol{C}_j - \boldsymbol{A}_j X_{j-1})^{-1} (\boldsymbol{A}_j Y_j + F_j), \quad j = 2(1)N_y$$

Here $X_1 = E$, v, $W_{\overline{N}+1} = (E - X_{\overline{N}})^{-1} Y_{\overline{N}}$, $(\overline{N} = N_y)$,

where $E = \begin{pmatrix} [1,0,...,0] & 0 \\ 0 & [1,0,...,0] \end{pmatrix}$.

The inverse matrix of $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ is

$B = A^{-1}, (BA = AB = E)$, $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, where

$$B_4 = (A_4 - A_3 A_1^{-1} A_2)^{-1}, \quad B_2 = -A_1^{-1} A_2 B_4,$$

$$B_1 = (A_1 - A_2 A_4^{-1} A_3)^{-1}, \; B_3 = -A_4^{-1} A_3 B_1.$$

The others operations with circular matrices and vectors of the second order can be easy obtain [6].

Results and discussion

1. Approbation of numerical algorithms

The special solution in the form $c_1(x,y,z) = g_1(z)\cos(\pi y/L)\sin(2\pi x/l)$, $c_2(x,y,z) = g_2(z)\cos(\pi y/L)\sin(2\pi x/l)$ of the PDE (1.1) was designed, where functions $g_1(z)$ $g_2(z)$ was the solution of the following boundary value problem for two ODE (for boundary condition (1.3)):

$$g_1''(z) - a_1^2 g_1(z) = 0, \; g_1(0) = 0, \; g_2''(z) - a_2^2 g_2(z) = 0$$
$$, g_2(Z) = 1, \; g_1(H_1) = g_2(H_1), \quad (7.1)$$
$$D_{1z} g_1'(H_1) = D_{2z} g_2'(H_1), \text{ where}$$

$$a_1 = \pi \sqrt{\left(\frac{4D_{1x}}{l^2} + \frac{D_{1y}}{L^2}\right) / D_{1z}},$$

$$a_2 = \pi \sqrt{\left(\frac{4D_{2x}}{l^2} + \frac{D_{2y}}{L^2}\right) / D_{2z}}.$$

The analytical solution of boundary-value problem of ODEs (7.1) is [5]:

$$g_1(z) = P_1 \sinh(a_1 z), \quad (7.2)$$

$g_2(z) = P_2 \cosh(a_2 z) + P_3 \sinh(a_2 z)$, where the constants are:

$$P_1 = \frac{P_2 \cosh(a_2 H_1) + P_3 \sinh(a_2 H_1)}{\sinh(a_1 H_1)},$$

$$P_2 = \frac{1 - P_3 \sinh(a_2 Z)}{\cosh(a_2 Z)},$$

$$P_3 = \frac{\cosh((a_1 - a_2)H_1)a_1 D_{1z}}{\sinh(a_2 H_2 + a_1 H_1)a_2 D_{2z}} + \frac{\sinh(a_1 H_1)\sinh(a_2 H_2)(a_1 D_{1z} - a_2 D_{2z})}{\sinh(a_2 H_2)\cosh(a_1 H_1)(a_1 D_{1z} - a_2 D_{2z})}.$$

Based on the literature source [4], it can be proofed that the solution to the BVP (7.1), obtained by hyperbolic spline function, coincides with the analytical (exact) solution (7.2) of the BVP (7.1).

2. Some numerical results

Measurements of peat samples for the determination of heavy metals – iron and calcium concentrations were carried out in the swamp of the Vilani municipality Knavu swamp. Peat analyses have been performed with the OPTIMA 2100 MS ICP/OES Spectrometer of the inductively associated plasma optic emission spectrometer of the Perkin Elmer firm in the laboratories of the Geotechnology and Eco-Industrial Research Centre of Rezekne Academy of Technologies [8]. We consider the metal concentration in the 2 layered peat blocks Ω with following measure:

$L = l = 1m$, $H_1 = 1m$, $H_2 = 1.5m$,

$Z = H_1 + H_2 = 2.5m$.

On the top of the earth $(z = Z)$ we have the measured concentration $c[mg/kg] \times 1000$ of calcium (Ca) in the following points in the (x, y) plane:

$c(0.1,0.2) = 0.62$, $c(0.5,0.2) = 0.57$,
$c(0.9,0.2) = 0.43$, $c(0.1,0.8) = 0.51$,
$c(0.5,0.5) = 0.43$, $c(0.9,0.5) = 0.64$,
$c(0.1,0.5) = 0.58$, $c(0.5,0.8) = 0.44$,
$c(0.9,0.8) = 0.72$.

This date are smoothing by 2D interpolation with MATLAB operator, using the spline function. We use following diffusion coefficients in the layers:

$$D_{1z} = 10^{-3},\ D_{2z} = 5 \cdot 10^{-4},\ D_{1x} = D_{2x} = 10^{-4},$$

$D_{1y} = D_{2y} = 10^{-5}$. We can see the distribution of concentration c in the (x, y) plane for Ca at $z = H_1$ for hyperbolic (Fig. 1) and parabolic (Fig. 2) spline.

We can see the distribution of concentration c in the (x, y) plane for Ca at $z = H_1$ for hyperbolic (Fig. 1) and parabolic (Fig. 2) spline, in the Fig. 3 – the averaged concentration C_2 within the second layer for hyperbolic spline, in the Fig. 4 – concentration's c curves at $y = L/2$ for hyperbolic spline.

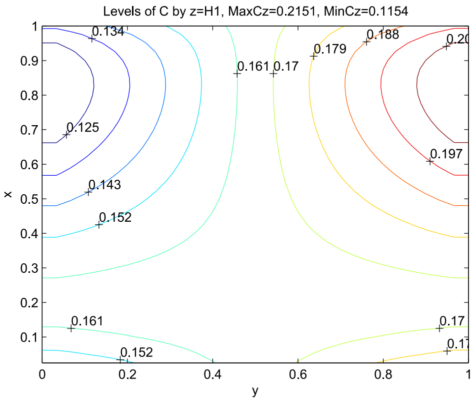

Fig. 1. Levels of c at $z = H_1$ for hyperbolic spline.

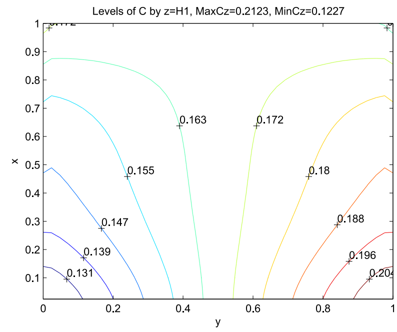

Fig. 2. Levels of c at $z = H_1$ for parabolic spline.

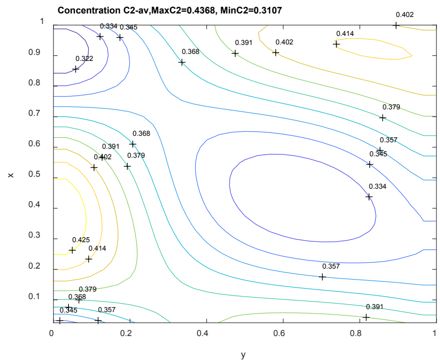

Fig. 3.The averaged concentration C_2 within the second layer for hyperbolic spline.

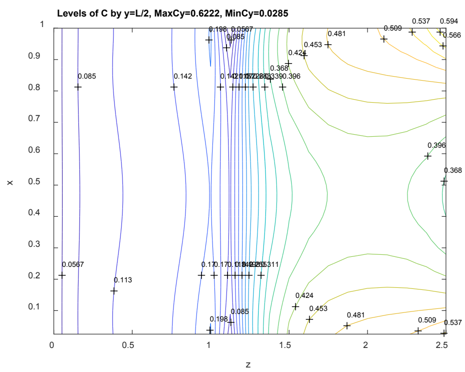

Fig. 4. Concentration's c curves at $y = L/2$ for hyperbolic spline.

Conclusions

1. The 3D diffusion problem in N layered domain described by a boundary value problem of the system of PDEs with piece-wise constant diffusion coefficients are approximated on the boundary value problem of a system of N PDEs. The last mentioned system is solved due to the finite difference method.
2. For reducing the 3D diffusion problem to 2D boundary value problem of a system of PDEs the conservative averaging method (CAM) along the vertical z-coordinate by using parabolic type splines and newly designed special hyperbolic and type splines is studied.
3. The calculation process compared the effectiveness of the parabolic and hyperbolic splines usage for 3D diffusion problem reduction to 2D boundary value problem and resulted in higher accuracy of the approximated solution directly with hyperbolic type splines.
4. A Test example has been created allowed a comparison of the analytical (exact) solution of the 1-D BVP with the numerically obtained solutions of parabolic type splines and hyperbolic type splines, to assess their accuracy. It was found that the solution obtained by hyperbolic splines functions coincided with the analytical solution of the 1-D problem under consideration. This indicates the usefulness of the further usage of the CAM for solving BVP problems with the hepl of hyperbolic type splines compared to the previously widely used parabolic type splines.
5. The theoretical and practical problem studied makes it possible to obtain an engineering algorithm for mathematical modelling mass transfer processes in multilayered domain.
6. The results of the numerical experiments can give some new physical conclusions about the distribution of metals concentration in different layered peat blocks.

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