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Deformation Transverse Shear Bending State of a Thin Plate Layer of an Anisotropic Geological Medium from the Action of Concentrated Energy Impulses

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Abstract **– A method is proposed for study the structural stability of the deformation state of structural blocks of the earth's crust, approximated in the form of plate layers of the geological medium when transverse shear bending from the action of concentrated energy impulses. Advances here are carried out in the two directions. First, in contrast to the previous article, the physical and mechanical model of the geological medium is endowed with anisotropic properties, which makes it possible to increase the adequacy of the obtained numerical results to the specifics of the real problem. Secondly, instead of the simplest bilinear 4-node finite elements, the special spectral non-algebraic 8-node finite iso-parametric finite elements are used, the use of which significantly increases both the accuracy of calculations and their reliability in the sense of ensuring the robustness of calculations for relatively small values of the plate thickness. It should be noted that the Finite Element Method uses exclusively only algebraic finite elements (power polynomials in the h-version and orthogonal polynomials in the p-version). It is known from approximation theory that the use of spectral non-algebraic approximations improves the quality of approximations. Therefore, their introduction into the structure of finite element calculations can improve the quality of modeling in the study of the strain-stress-state (SSS) of the geological medium. A structural block (SB) is understood as a plate layer with plan dimensions exceeding the thickness by more than 10 times. The identification of hazardous zones in the rock massive due to stress concentration is complemented by the development of mechanical, mathematical and computational tools for modeling the curvature of the earth's crust during bending based on the classical theory of Kirchhoff and refined Reissner-Mindlin theory. Test calculations have shown that the accuracy of the calculation and the quality of geometric modeling of fragments of an anisotropic geological environment based on the refined 8-node spectral finite element is significantly better than for the 8-node algebraic finite element.**

Keywords – stability of geological environment, radioactive waste, bending of a Reissner-Mindlin plate of medium thickness, finite element method.

I. INTRODUCTION

In our case, the structural block (SB), in which it is planned to place the radioactive waste, can be represented in the first approximation as a plate of finite thickness (with the ratio of its thickness to characteristic length less than 1/20) using the classical Kirchhoff model or the refined Reissner-Mindlin model [1] for plates with a relatively thicker thickness (with a ratio greater than 1/20, which are called medium-thickness plates). In addition, an analysis of the deformation of the surface upper layer is introduced, idealized using a refined (allowing to take into account the effect of a small parameter of relative thickness and transverse shear deformation) bending model for very thin plate layers with a ratio of the average size to thickness of about 10000. This leads to finite element modeling for very stiff computational schemes that require appropriate approaches to obtain a reliable solution. In this regard, the identification of hazardous zones in the rock mass due to the stress concentration caused by the nature of the change in SSS is complemented by the development of mechanical-mathematical and computational algorithms and modeling tools based on the theories of

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Kirchhoff and Reissner-Mindlin formulations [2].

II. RESULTS AND DISCUSSION

Spectral Shape Functions for an 8-Node Finite Element

In a rectangular Cartesian coordinate system xy, consider a curvilinear finite element (Fig. 1), all nodes of which are numbered from 1 to 8.

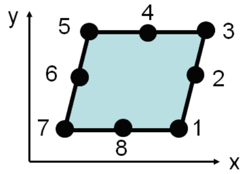

Fig.1. Iso-parametric 8-node finite element in the physical plane with the numbering of nodes.

In another rectangular Cartesian coordinate system *ξη,* we introduce an auxiliary square with side 2 (Fig. 2). On the sides of this square, select eight points in accordance with the number of nodes of the finite element.

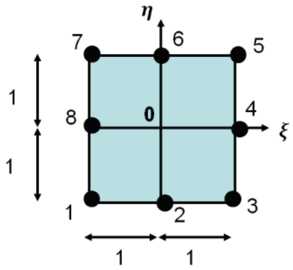

Fig. 2. Generating (Master) 8 – node element in the computational plane.

The constructed non-algebraic shape functions ψ_r (ξ, η) for r = (1, 8) take the form [11-13]

$$\psi_r = \psi_r^{(1)} + \psi_r^{(2)} \quad (r = \overline{1,8})$$

$$\psi_1^{(1)} = (1-t)(1-z), \qquad \psi_3^{(1)} = (t)(1-z),$$
$$\psi_5^{(1)} = (t)(z), \psi_7^{(1)} = (1-t)(z),$$
$$\psi_r^{(1)} = 0, (r = 2,4,6,8)$$
$$\psi_2^{(2)} = \chi_{11}(z) sin\pi t, \qquad \psi_4^{(2)} = \chi_{21}(t) sin\pi z,$$
$$\psi_6^{(2)} = \chi_{21}(z) sin\pi t,$$
$$\psi_8^{(2)} = \chi_{11}(t) sin\pi z,$$
$$\psi_1^{(2)} = -1/2(\psi_2^{(2)} + \psi_8^{(2)}), \psi_3^{(2)} = -1/2(\psi_2^{(2)} + \psi_4^{(2)}),$$
$$\psi_5^{(2)} = -1/2(\psi_4^{(2)} + \psi_6^{(2)}), \psi_7^{(2)} = -1/2(\psi_6^{(2)} + \psi_8^{(2)}),$$
$$\chi_{11}(z) = -e^{-\alpha} c_1 + e^{-\alpha z} c_{21}, \chi_{21}(z) = e^{-\alpha(1-z)} c_{21} + e^{-\alpha z} c_{11},$$
$$c_{11} = -e^{-\alpha} c_{21}, \; c_{21} = (1 - e^{-2\alpha} \alpha = 2\sqrt{2},$$
$$\alpha = 2\sqrt{2}, t = (1+\xi)/2, z = (1+\eta)/2 \qquad (1)$$

Non-algebraic solution method

A plate SB is understood as a layer with plan dimensions **a** exceeding the thickness **h** by more than 10 times. The calculation is based on the previously developed original problem-oriented finite element software complex together with a set of programs for modeling the SSS of a heterogeneous medium [5], supplemented by the methodology of energy analysis of stress concentration for the problems of geodynamic zoning for the model of the plane problem of the theory of elasticity (generalized plane stress state) [6]. In this case, the problem is solved for a thin plate layer and a layer of a plate of medium thickness, which is a transitional option for further approximation to the 3D formulation of the solution to the problem of predicting the stability of the SB. An analysis of our earlier calculations of hazard levels using four criteria for a plane problem showed that the determining factors here are not the stress values themselves (which for the elastic model at the tops of cracks and faults can take any large values), but the integral characteristics of the stress concentration. The scheme for solving these problems is reflected in [4, 6-7].

It is required to find the components of the stress strain tensors in the plate layer.

The results are, within the framework of the accuracy of the involved mechanical and mathematical modeling, recommendations are formed on the stability and safety of the insulating properties of the medium within the investigated SB.

Modeling the action of an energy impulse

The case of the impact of an instantaneous point energy impulse is described in detail in [1]. The superposition principle makes it possible to consider the solution of the problem of plate bending from the action of a point energy impulse under the simply supported boundary conditions. To simulate (within the framework of the formulation of the boundary value problem) the action of an instantaneous energy impulse on the plate, we will use the energy-force analogy due to the law of conservation of energy. For this, we first solve the following problem of plate bending due to the action of a unit concentrated force 1 (ξ, η) applied at a point (ξ, η).

As a result of the action of a concentrated unit force, the internal forces in the plate will produce work equal to the work of the applied external force on the displacement caused by it:

$$A_1 = 1/2 \times 1 \times \omega(\xi,\eta) \qquad (2)$$

This work, according to the law of conservation of energy, will transform into the potential energy of deformation of the plate

$$U_1 = A_1 \qquad (3)$$

Let an energy impulse with amplitude J (ξ, η) act at some point on the plate (ξ, η).

We introduce the ratio of energy quantities

$$\Lambda = J(\xi,\eta/U_1) \qquad (4)$$

Then the magnitude of the impulse will be expressed through the magnitude of the potential energy:

$$J(\xi,\eta/U_1) = \Lambda \times U_1 \qquad (5)$$

If the value of the concentrated force at the point (ξ, η) is equal to Λ, then the value of the accumulated potential energy of deformation will be U_Λ:

$$U_\Lambda = \Lambda \;\; U_1 \qquad (6)$$

From (2) - (6) it follows that the concentrated force of the quantity Λ at the point (ξ, η) accumulates in the plate an energy equal to the energy impulse acting at the same point. As a result, the stress-strain state of the plate is found from the action of a point energy (for example, seismic) impulse. Thus, an energy-force analogy is realized, which allows simulating concentrated energy impulses using concentrated shear forces during bending of simply supported polygonal plates [1].

Refined Reissner-Mindlin theory for anisotropic plates

The initial formulation of the problem of bending of medium and small thickness h under the action of transverse loads and instantaneous energy impulse with Reissner-Mindlin kinematics, which correspond to the occurrence of vertical displacement ω and two rotation angles $\theta_i\,(i = 1{,}2)$ of the normal to the original undeformed median plane of the plate: $\{\omega, \theta_i\}$ is reflected in [2]. In the refined Reissner-Mindlin theory [2], three independent degrees of freedom are used: ω - deflection; θ_x , θ_y – rotations angles. Here: $\{w, \theta_i\,(i = 1, 2)\} = \{w, \theta_x, \theta_y\}$ are the required field functions.

According to the Reissner-Mindlin formulation, the displacements are:

$$u_x = z\theta_x(x, y), u_y = z\theta_y(x, y), w = w(x, y) \quad (7)$$

Represent field functions by using shape functions

$$w = \sum_{r=1}^{8} \psi_r(\xi, \eta) w_r,\ \theta_x = \sum_{r=1}^{8} \psi_r(\xi, \eta)\theta_{xr},\ \theta_y = \sum_{r=1}^{8} \psi_r(\xi, \eta)\theta_{yr}, \quad (8)$$

Here w_r , θ_{xr} , θ_{yr} – degrees of freedom. Using the iso-parametric approach, we associate each point of the computational square (Fig. 2) with some point of the original finite element (Fig. 1), putting

$$x = \sum_{r=1}^{8} \psi_r(\xi, \eta) x_r,\ y = \sum_{r=1}^{8} \psi_r(\xi, \eta) y_r, \quad (9)$$

where x_r, y_r are corresponding coordinates of the nodes.

A typical block of the stiffness matrix of the element $[K_{rs}^{e}]$ will be written as the sum of the bending and shear parts [11-13]

$$[K_{rs}^{e}] = [K_{rs}^{e}]b + [K_{rs}^{e}]s$$

$$[K_{rs}^{e}]b = \int_{-1}^{1}\int_{-1}^{1} D_1\,[\beta_r]b^T[\kappa]b[\beta_s]b\,|[J(\xi, \eta)]|\,d\xi d\eta$$

$$[\beta r]_b = \begin{bmatrix} 0 & \partial\psi_r/\partial x & 0 \\ 0 & 0 & \partial\psi_r/\partial y \\ 0 & \partial\psi_r/\partial y & \partial\psi_r/\partial x \end{bmatrix}$$

$$[\beta r]_s = \begin{bmatrix} \partial\psi_r/\partial x & \psi_r & 0 \\ \partial\psi_r/\partial y & 0 & \psi_r \end{bmatrix}$$

$$[\kappa] = \begin{bmatrix} 1 & \mu_2 & 0 \\ \delta_2\mu_1 & \delta_2 & 0 \\ 0 & 0 & \delta_G(1 - \mu_1\mu_2) \end{bmatrix}, \quad (9)$$

where

$$D_1 = \frac{E_1 h^3}{12(1 - \mu_1\mu_2)}$$

$$\delta_2 = \frac{E_2}{E_1},$$

$$\delta_G = \frac{G}{E_1},$$

$$\mu_2 = \delta_2\mu_1 \quad (10)$$

Here E_1, E_2 – young's moduli for tension-compression in the main directions *x, y* [14]; G – shear modulus; μ_1 – poisson's ratio, characterizing the contraction in the *y* direction when stretching-compressing in the *x* direction; *h* is the thickness of the plate; | [J (ξ, η)] | - modulus of the determinant of the Jacobi matrix [2]; the top "T" symbol stands for transposition.

The shape functions for the corner points of the constructed eight-node finite element are represented by the sum of bilinear and non-polynomial terms (for intermediate nodes, there are only non-polynomial functions). However, it is known [2] that a bilinear finite element can lead to "locking" when the plate becomes thinner due to a strong shear stiffening of the resolving system of finite element equations. Therefore, it seems appropriate to express the shear part of the stiffness matrix (9) as a sum of four terms, the first three of which contain integrals of bilinear functions or their products by non-polynomial functions, and the last term is an integral of only a non-polynomial function:

$$[K_{rs}]s = \sum_{i=1}^{2}\sum_{j=1}^{2} \delta_G E_1 \int_{-1}^{1}\int_{-1}^{1} h[\beta_r^{(i)}][\beta_s^{(j)}]^T |[J]|\, d\xi_i d\eta_j$$

$$\begin{bmatrix} \frac{\partial\psi_r^k}{\partial x} & \psi_r^{(k)} & 0 \\ \frac{\partial\psi_r^k}{\partial y} & 0 & \psi_r^{(k)} \end{bmatrix} \quad (11)$$

The fundamental difference between the obtained representation (11) and the original (9) lies in the important possibility to perform integration for each of the four terms separately - according to its specially constructed quadrature formulas.

The variation formulation of the problem consists in finding a solution that satisfies the conditions of equality to zero of the first variation of the total potential energy of the plate and the strict positivity of the second variation:

$$\delta\Pi = 0;\ \delta^2\Pi > 0 \quad (12)$$

Here P = U + A is the total potential energy of the plate, U is the potential energy of deformation, A is the work of external forces.

The calculation of anisotropic plates will be carried out using the example of square plates with homogeneous (Fig. 3, a) and mixed (Fig. 3, b) boundary conditions.

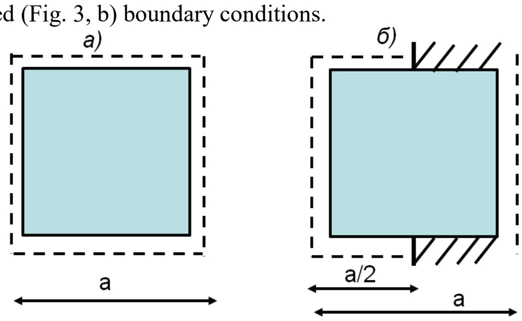

Fig. 3. Plates with different boundary conditions:
a) a simply supported plate; b) mixed: partially supported.

Results of finite element analysis of an anisotropic Reissner-Mindlin plates

The calculation of thin Reissner-Mindlin plates by the finite element method causes significant computational difficulties associated with the rigidity (stiffening) of the resolving system of linear algebraic equations and leading to the "locking" of the sought solution at the displacement and potential energy levels [2] close to zero.

The following conclusions can be drawn that with an increase in the number of finite elements:
1) For plates of medium thickness, the values of the solution are obtained, although they differ from the desired deflections, but do not tend to zero and partially reflect the change [2] of the exact solution depending on the thickness of the plate.
2) With a further decrease in thickness, solution values are obtained that rapidly tend to zero.

The general conclusion here is that exact integration of stiff matrix for the finite elements of the Reissner-Mindlin plates does not allow obtaining satisfactory values for the desired solution in the case of sufficiently thin plates. In order to improve the quality of the solution, finite element calculations were performed on the same mesh using reduced (non-exact) integration. The calculation results show that the reduced integration scheme, in contrast to the exact integration scheme, gives rather good results approaching the exact value on the meshes. It is essential that in this case it is possible to take into account the effect of thickness on the parameters of plate deformation. However, such a scheme is not robust: for weakly coupled boundaries, it leads to zero energy modes, to eliminate which stabilizing finite elements were built [2, 8-10].

Finally, in order to further increase the accuracy of finite element calculations, we use a selective-reduced integration scheme with decomposition [11], where the bending part of the potential energy is integrated according to the 4-point Gauss-Legendre quadrature formula (“exact integration”), and the shear part - according to the one-point quadrature formula (“approximate integration”) with the decomposition of shear stiff matrix [11-13].

Numerical experiments for an 8-node spectral finite element confirmed the reliability of the proposed decomposition of the shear stiffness matrix and revealed the following effective numerical integration scheme: for the first three integrals in (11), one should use the one-point Gauss-Legendre formula (1x1), and for the latter, the four-point (2x2). Since the application of such a procedure for the bending part of the stiffness matrix did not lead to practical results, the traditional integration over nine (3x3) Gauss-Legendre points was used there. Attempts to construct other numerical integration schemes for the considered non-algebraic finite element were unsuccessful: the element was "locked" with the plate refinement (the proposed procedure turned out to be completely useless for an eight-node quadratic algebraic serendipity finite element due to the fundamental impossibility of such a division in polynomial shape functions).

This is illustrated by the results for composite plate layers of an anisotropic layered material composed of orthogonally oriented fibers with mechanical parameters corresponding to the properties structure of a composite type of glass fibers impregnated with resin with the following stiffness parameters [14]: $E_1 = 1.6 \times 10^{11}$ Pa, $E_2 = 2.6 \times 10^{11}$ Pa, G= 0.42×10^{11} Pa.

Figure 4 shows curves characterizing the value of the maximum dimensionless deflection $W_C/(\frac{Pa^2}{D_1})$ of a square, simply supported anisotropic plate, which is under the action of a concentrated energy impulse P, depending on the parameter of the relative thickness. The dotted lines represent the solutions obtained on a 4 x 4 finite element mesh, and the solid lines represent the 6 x 6 mesh.

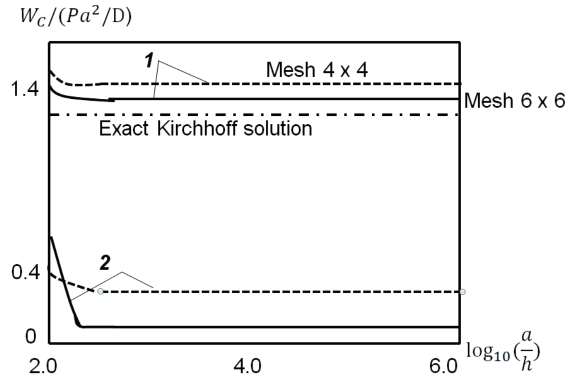

Fig. 4. Simply supported plate (Fig. 3a): 1 - selectively reduced integration with decomposition stiff matrix; 2 - exact integration.

Number 1 corresponds to the use of the proposed scheme of selectively reduced integration with preliminary decomposition of the shear part of the stiffness matrix, and number 2 corresponds to the solutions obtained using the well-known integration scheme [2]: 3 x 3 points for the bending part and 2 x 2 points for the shear part stiffness matrices. The dash-dotted line here corresponds to the value of the exact solution for the classical Kirchhoff theory of bending as applied to a simply supported square plate, obtained using double Fourier sine series and having the next form:

$$\frac{(w_C)}{Pa^2/D_1} = \frac{4}{\pi^4} \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=1,3,5,\dots}^{\infty} (-1)^{m-1}(-1)^{n-1} \frac{1}{(m^4 + 2\delta_3 m^2 n^2 + \delta_2 n^4)}$$

$$\delta_3 = \mu_2 + 2\delta_G(1 - \mu_1\mu_2$$

Fig. 5 reflects the dependence of the dimensionless central deflection for a plate with mixed (discontinuous) boundary conditions (Fig. 3b): when plate is clamped along the halves of two opposite sides and simply support on the remaining sections of the boundary. Here, the solid line corresponds to the use of the generated non-polynomial shape functions for the proposed scheme of selectively reduced integration with preliminary decomposition of the shear part of the stiffness matrix, and the dashed line to the use of an eight-node algebraic serendipity finite element with the known scheme of selectively reduced integration. The results were obtained on a 6 x 6 finite element mesh and are similar to the previous results (Fig. 3). For a quadratic algebraic serendipity finite element, usually effective selectively reduced integration does not exclude shear “locking” for very thin plates with a complex nature of the boundary conditions, what is consistent with [2]. The dot-and-dash line presents the solution obtained by the analytical method of dual equations for Kirchhoff model [14].

Comparison of the above results testifies to the efficiency of using the constructed 8-node non-polynomial finite element based on the proposed selective-reduced scheme with preliminary decomposition of the shear part of the stiffness matrix and the implementation of the possibility of adjustable selection of quadrature points in numerical integration.

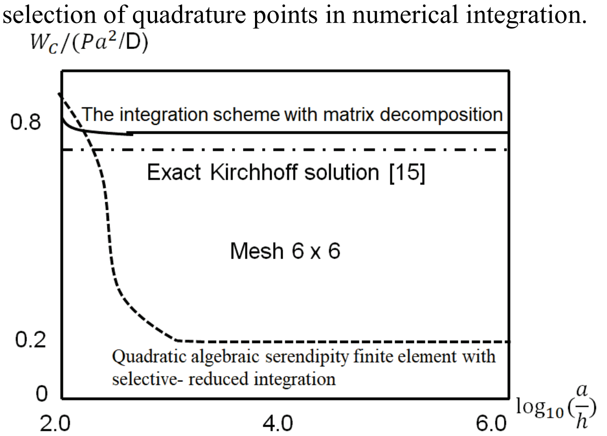

Fig. 5. Partially supported plate: Fig. 3b.

III. Conclusion

As a result of the research:

- effective refined finite elements of the Reissner-Mindlin plates under the action of an instantaneous energy impulse have been implemented, which expand the capabilities of the previously created software for calculating the stress-strain state of a heterogeneous anisotropic geological medium and can form the basis of the new software packages for applications;
- computational finite element tools for calculating very thin plates and nano-films (the ratio of the side length of the plate to its thickness of the order of: ten thousand, one hundred thousand and one million) in bending;
- the accuracy of the calculation and the quality of geometric modeling of complex fragments of the geological medium on the basis of an 8-node spectral non-algebraic finite element have been significantly increased.

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