FLOW OF INCOMPRESSIBLE VISCOUS FLUID CAUSED BY THE MOTION OF INFINITE CONE OF ROTATION ALONG THE AXIS OF SYMMETRY

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Abstract. *In this paper, we solve the problem of perturbation of an incompressible viscous fluid caused by the motion of an infinite cone in the direction of the axis of symmetry. The method involves the introduction of a harmonic current function. The Navier-Stokes equations with the incompressibility equation reduces to a system of ordinary differential equations.*
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Introduction

Let us consider the motion of an infinite cone of revolution in the direction of the axis of symmetry. Let U speed of movement of the cone, 2α Is the angle at its vertex. The problem consists in finding the velocity field characterizing the perturbation of an incompressible viscous fluid, caused by the motion of the cone.

There are many approaches to solving boundary-value problems for the Navier-Stokes equations. Two-dimensional problems are considered on a complex field. In spatial problems, the solution is sought in a previously undefined form. The success of such a search largely depends on the professionalism of the researcher. The numerical solution has an applied meaning. [3, 4]

Description of the method

For the incompressible viscous fluid, the Navier-Stokes system of equations has the form:

$$\begin{gathered} \frac{\partial \vec{\upsilon}}{\partial t} = -(\vec{\upsilon} \cdot \nabla)\vec{\upsilon} + \nu \Delta \vec{\upsilon} - \frac{1}{\rho} \nabla p + \vec{f} \, , \\ \nabla \cdot \vec{\upsilon} = 0 \, , \end{gathered} \tag{1}$$

where the second equation is called the incompressibility condition. From equation (1) N.E. Cochin obtained an equation, later called the generalized Helmholtz equation [1, p.404]:

$$\frac{\partial \vec{\Omega}}{\partial t} + (\vec{\upsilon} \cdot \nabla)\vec{\Omega} - (\vec{\Omega} \cdot \nabla)\vec{\upsilon} = \nu \Delta \vec{\Omega} \, , \tag{2}$$

where $\vec{\Omega} = \nabla \times \vec{\upsilon}$.

We satisfy the equation of incompressibility by introducing the vector stream function:

$$\vec{\upsilon} = \nabla \times \vec{\Psi} \, . \tag{3}$$

Such an action will make sense in those cases when at least two components of the velocity vector are not identically equal to zero. In other conditions, it is necessary to solve the task in another way.

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Let us consider in detail the function of the current $\vec{\Psi}$ Is determined to within $\nabla\varphi$, where φ - arbitrary scalar function, it can always be assumed that

$$\nabla\vec{\Psi} = 0 . \tag{4}$$

Really

$$\vec{\Psi} = \vec{\Psi}_1 + \nabla\varphi; \ \nabla\vec{\Psi} = \nabla\vec{\Psi}_1 + \Delta\varphi,$$

choosing φ so that $\Delta\varphi = -\nabla\vec{\Psi}_1$ obtain (4).
In view of what has been said

$$\vec{\Omega} = \nabla\times\vec{\upsilon} = \nabla\times\nabla\times\vec{\Psi} = \nabla(\nabla\cdot\vec{\Psi}) - \Delta\vec{\Psi} ,$$

Considering (4)

$$\vec{\Omega} = -\Delta\vec{\Psi} . \tag{5}$$

Now, if we assume that $\vec{\Psi}$ - harmonic function,

$$\Delta\vec{\Psi} = 0 , \tag{6}$$

Then the generalized Helmholtz equation (2) will be satisfied. If we find a solution of the Laplace equation for a vector stream function that satisfies the boundary conditions of the hydrodynamic problem of incompressible viscous fluid, then the velocity field is found from (3). From (6), after simple transformations, we have

$$\nabla\times(\Delta\vec{\Psi}) = \Delta\vec{\upsilon} = 0 . \tag{7}$$

We form the boundary conditions (external problem)

$$\begin{cases} \nabla\times\vec{\Psi}\Big|_{\partial\Lambda} = \vec{u}(P), P\in\partial\Lambda, \\ \nabla\times\vec{\Psi} \xrightarrow[r\to\infty]{} \vec{U}. \end{cases} \tag{8}$$

where Λ - the area filled with fluid, $\partial\Lambda$ - he boundary of this region, $\vec{u}(P)$ - Is the velocity of the liquid at the boundary of the surface under consideration, $\vec{U}$ - Is the velocity of a steady flow of a liquid.

In view of the uniqueness theorem for the solution of boundary value problems (8) for the system of partial differential equations (6), the solution obtained in the form (3) is unique.

The solution of the problem

Consider a spherical coordinate system with origin at the vertex of the cone. Then the boundary conditions have the form:

$$\upsilon_r\Big|_{\theta=\alpha} = -U\cos\alpha; \ \upsilon_\theta\Big|_{\theta=\alpha} = U\sin\alpha; \ \upsilon_\varphi\Big|_{\theta=\alpha} = 0 . \tag{9}$$

Because the equation is symmetric in $\varphi : \dfrac{\partial}{\partial\varphi} = 0$; $\upsilon_\varphi \equiv 0$.

Assuming $\vec{\Psi}(0, 0, \Psi_\varphi(r, \theta))$, we obtain from (17) the velocity components (the first two components are zero to satisfy $\upsilon_\varphi \equiv 0$):

$$\begin{aligned} \upsilon_r &= \frac{1}{r}\frac{\partial \Psi_\varphi}{\partial \theta} + \frac{\cot\theta}{r}\Psi_\varphi; \\ \upsilon_\varphi &= -\frac{\partial \Psi_\varphi}{\partial r} - \frac{\Psi_\varphi}{r}. \end{aligned} \tag{10}$$

Since the boundary conditions do not contain r, we seek solutions in the form

$$\Psi_\varphi = r \cdot f(\theta). \tag{11}$$

Substituting (11) into (9) we obtain:

$$f'' + f' \cot\theta + 2f = 0. \tag{12}$$

Thus, the problem is reduced to the search for a solution of a second-order linear differential equation. By means of the theory of ordinary differential equation we obtain the following result:

$$f(\theta) = A\cos\theta + B\left(1 - \cos\theta \cdot \ln\cot\frac{\theta}{2}\right). \tag{13}$$

Then satisfying (9), we obtain:

$$\begin{aligned} A &= -\frac{U}{2}\sin\alpha \ln\cot\frac{\alpha}{2}; \\ B &= -\frac{U}{2}\sin\alpha. \end{aligned} \tag{14}$$

In this way $f(\theta)$ Is determined from the relations (13) and (14).

Let us find the velocity components:

$$\begin{aligned} \upsilon_r &= f' + f\cot\theta = \frac{1}{\sin\theta}\left[A\cos 2\theta + 2B\cos\theta - B\cos 2\theta \ln\cot\frac{\theta}{2}\right]; \\ \upsilon_\theta &= -2f = -2A\cos\theta - 2B\left(1 - \cos\theta \ln\cot\frac{\theta}{2}\right); \end{aligned} \tag{15}$$

$$\begin{aligned}
\upsilon_r &= \frac{U}{2}\frac{\sin\alpha}{\sin\theta}\left[-\cos 2\theta \ln\cot\frac{\alpha}{2} - 2\cos\theta + \cos 2\theta \ln\cot\frac{\theta}{2}\right];\\
\upsilon_\theta &= U\sin\theta\left[\cos\theta\ln\cot\frac{\alpha}{2} - \cos\theta\ln\cot\frac{\theta}{2} + 1\right];\\
\upsilon_\varphi &= 0.
\end{aligned} \tag{16}$$

Thus, the velocity field is determined by the relations (16) with, which is the solution of the initial problem.

Conclusions

The main advantage of this method is that the problem reduces to solving a system of linear partial differential equations, and from theory it is known that a solution always exists, and the only one. Very often such a system does not have solutions in elementary functions, or, if it does, obtaining such expressions strongly depends on the professionalism of the decider. In any case, this is a strong simplification, and indeed, the indicated problem (3), (6), (8) from the point of view of approximate calculations has less complexity.

Attachment

To facilitate the reading of the article, the expressions for differential operators in spherical coordinates are given below. Components of the vector Laplace operator of the function [2]:

$$\begin{cases}
(\Delta\vec{\Psi})_r = \Delta\Psi_r - \dfrac{2\Psi_r}{r^2} - \dfrac{2}{r^2}\dfrac{\partial\Psi_\theta}{\partial\theta} - \dfrac{2\Psi_\theta}{r^2}\cot\theta - \dfrac{2}{r^2\sin\theta}\dfrac{\partial\Psi_\varphi}{\partial\varphi};\\
(\Delta\vec{\Psi})_\theta = \Delta\Psi_\theta + \dfrac{2}{r^2}\dfrac{\partial\Psi_r}{\partial\theta} - \dfrac{\Psi_\theta}{r^2\sin^2\theta} - \dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial\Psi_\varphi}{\partial\varphi};\\
(\Delta\vec{\Psi})_\varphi = \Delta\Psi_\varphi - \dfrac{\Psi_\varphi}{r^2\sin^2\theta} + \dfrac{2}{r^2\sin^2\theta}\dfrac{\partial\Psi_r}{\partial\varphi} + \dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial\Psi_\theta}{\partial\varphi};
\end{cases} \tag{17}$$

where $\Delta A = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial A}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial A}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 A}{\partial\varphi^2}$

And the components of the velocity vector $\vec{\upsilon} = \nabla\times\vec{\Psi}$ have the form:

$$\begin{cases}
\upsilon_r = \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial\theta}\left(\Psi_\varphi\sin\theta\right) - \dfrac{\partial\Psi_\theta}{\partial\varphi}\right]\\
\upsilon_\theta = \dfrac{1}{r\sin\theta}\dfrac{\partial\Psi_r}{\partial\varphi} - \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\Psi_\varphi\right)\\
\upsilon_\varphi = \dfrac{1}{r}\left[\dfrac{\partial}{\partial r}\left(r\Psi_\theta\right) - \dfrac{\partial\Psi_r}{\partial\theta}\right]
\end{cases} \tag{18}$$

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